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# On the Construction of Zero Energy States in Supersymmetric Matrix Models

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## Abstract

For the SU(N) invariant supersymmetric matrix model related to membranes in 4 space-time dimensions, the general solution to the equation(s) $Q^\dagger\Psi = 0$ ($Q\chi = 0$) is determined for N odd. For any such (bosonic) solution of $Q^\dagger\Psi = 0$, a (fermionic) $\Phi$ is found that (formally) satisfies $Q^\dagger\Phi = \Psi$.
For the analogous model in 11 dimensions the solution of $Q^\dagger\Psi = 0(Q\Psi = 0)$ is outlined.

Previous methods to study the existence of zero energy bound states in SU(N)-invariant supersymmetric matrix models do not lead to actual solutions of $(Q^\dagger Q + QQ^\dagger)\Psi = 0$. It therefore seems valuable to propose a different route. For the model corresponding to 4 space time dimensions, I solve the equation(s) $Q^\dagger\Psi = 0$ ($Q\chi = 0$) and for any such $\Psi \,\epsilon\, \mathcal{H}_+$ (= the space of bosonic, SU(N) invariant wavefunctions) determine a $\Phi$ satisfying $Q^\dagger\Phi = \Psi$. By proving that $\Phi$ is normalizable, provided $\Psi$ is, one could (as observed by J. Fröhlich) prove rigorously the absence of a (bosonic) groundstate in this model, for arbitrary (odd) N. In cases when a ground state does exist (as is expected, e.g., for the 11-dimensional model; see [1-9] for some literature), comparing the general solution of $Q^\dagger\Psi = 0$ with the general solution of $Q\chi = 0$ (or proving $\Phi$ to be non-normalizable for some normalizable $\Psi = Q^\dagger\Phi$) will yield the explicit construction of the ground state wave-function. For the supersymmetric matrix model corresponding to membranes in 11 space-time dimensions, the calculation is set up in a form where the determination of the expected ground state seems feasible.

Let

$$\begin{aligned} Q &= 2\partial_a \frac{\partial}{\partial\lambda_a} + iq_a\lambda_a \\ Q^\dagger &= -2\overline{\partial}_a\lambda_a - iq_a\frac{\partial}{\partial\lambda_a} \end{aligned} \tag{1}$$

where $\partial_a = \frac{\partial}{\partial z_a}$, $z_a \,\epsilon\, \mathbb{C}$, $q_a = \frac{i}{2} f_{abc} z_b \overline{z}_c$ ($f_{abc}$ being totally antisymmetric, real, structure constants of SU(N)) and $\lambda_a$ ($\frac{\partial}{\partial\lambda_a}$) being fermionic creation (annihilation) operators satisfying $\{\lambda_a, \frac{\partial}{\partial\lambda_b}\} = \delta_{ab}$ , $\{\lambda_a, \lambda_b\} = 0 = \{\frac{\partial}{\partial\lambda_a} , \frac{\partial}{\partial\lambda_b}\}$. The hamiltonian of the model is

$$QQ^\dagger + Q^\dagger Q = -4\partial_a\overline{\partial}_a \;+\; q^2 + f_{abc}z_c\lambda_a\lambda_b + f_{abc}\overline{z}_c\frac{\partial}{\partial\lambda_b}\frac{\partial}{\partial\lambda_a}. \tag{2}$$

$Q$ and $Q^\dagger$ commute with the operators of SU(N),

$$J_a \;:=\; -if_{abc}(z_b\partial_c + \overline{z}_b\overline{\partial}_c + \lambda_b\partial_{\lambda c}) \tag{3}$$

and on the Hilberspace $\mathcal{H}$ of gauge-invariant states, $Q^2 = i\overline{z}_a J_a = 0$. Let

$$\Psi = \sum_{j=0}^{N^2-1} \frac{1}{j!}\,\psi_{a_1\cdots a_{2j}}\,\lambda_{a_1}\cdots\lambda_{a_{2j}} \tag{4}$$

(an analogous discussion could be applied to states in $\mathcal{H}_-$, i.e. states containing only odd numbers of $\lambda$'s). The equations $Q\chi = 0, Q^\dagger\Psi = 0$ then read

$$i(2k-1)\; q_{[a_1}\chi_{a_2\cdots a_{2k-1}]} \;=\; 2\partial_a\chi_{a_1\cdots a_{2k-1}a} \tag{5}$$

$$(2k-1)2\overline{\partial}_{[a_1}\psi_{a_2\cdots a_{2k-1}]} \;=\; iq_a\psi_{a_1\cdots a_{2k-1}a} \tag{6}$$

where $k = 1, \cdots, K := \frac{N^2-1}{2}$. Think of (5) as equations for $\chi^{(2k-2)} = \{\chi_{a_1\cdots a_{2k-2}}\}$, provided $\chi^{(2k)} = \{\chi_{a_2\cdots a_{2k}}\}$ is known; it is not difficult to verify that

$$\chi^{[in]}_{a_1\cdots a_{2k-2}} := \frac{-2i}{q^2} q_{a_{2k-1}}\partial_{a_{2k}}\chi_{a_1\cdots a_{2k}} \tag{7}$$

solves (5). At each stage of the interaction (eventually leaving only one single free function, $\chi_{a_1\cdots a_{2k}} = \epsilon_{a_1\cdots a_{2k}}\,\tilde{\chi}$) a solution of the homogeneous equation, i.e.

$$q_{[a}\chi^{[h]}_{a_1\cdots a_{2k-2}]} \;\equiv\; 0 \tag{8}$$

may be added. Similarly, one may verify that

$$\psi^{(in)}_{a_1}\cdots_{a_{2k}} := \frac{(2k)(2k-1)}{q^2}\ 2i\ q_{[a_1}\overline{\partial}_{a_2}\psi_{a_3}\cdots_{a_{2k}]} \tag{9}$$

(determining $\psi^{(2k)}$ in terms of $\psi^{(2k-2)}$) solves (6), so that the *general* solution of (6) is given by

$$\Psi \ = \ \Psi^{(in)} \ \oplus \ \Psi^{(h)}, \tag{10}$$

with

$$q_{a_{2k}}\ \psi^{(h)}_{a_1}\cdots_{a_{2k}} \ \equiv \ 0. \tag{11}$$

With the direct sum property indicated in (10) ($\int \psi^{(h)\star}_{a_1}\cdots_{a_{2k}}\ \psi^{(in)}_{a_1}\cdots_{a_{2k}} = 0 = \int \psi^{[in]\star}_{a_1}\cdots_{a_{2h}}\psi^{[h]}_{a_1}\cdots_{a_{2k}}$) the choice of the particular solution(s) of the inhomogenous equation(s) is canonical. Note that

$$(q\partial)\psi_{a_1}\cdots_{a_n} \ + \ n\psi_{a[a_2}\cdots_{a_n}\partial_{a_1]}q_a = 0 \tag{12}$$

when $J_a\Psi = 0$. Now define

$$\Phi = \sum_{k=1}^{K}\frac{1}{(2k-1)!}\ \phi_{a_1}\cdots_{a_{2k-1}},\lambda_{a_1}\cdots\lambda_{a_{2k-1}}$$

by

$$\phi_{a_1}\cdots_{a_{2k-1}} \ := \ \frac{-i}{q^2}\ (2k-1)q\ [_{a_1}(\psi_{a_2}\cdots_{a_{2k-1}]} - 2\ (2k-2)\ \overline{\partial}_{a_2}\phi_{a_3}\cdots_{a_{2k-1}}]) \tag{13}$$

. Then $-Q^\dagger\Phi = \Psi$ , i.e.

$$\begin{aligned} iq_a\phi_a \ &= \ \psi \\ iq_c\phi_{abc} \ + \ 4\ \overline{\partial}_{[a}\phi_{b]} \ &= \ \psi_{ab} \qquad (14) \\ &\ \vdots \\ iq_{a_{2k+1}}\phi_{a_1}\cdots_{a_{2k+1}} \ + \ (2k)2\overline{\partial}_{[a_1}\phi_{a_2}\cdots_{a_{2k}]} \ &= \ \psi_{a_1}\cdots_{a_{2k}} \\ &\ \vdots \\ \psi_{a_1}\cdots_{a_{2k}} = (2K)2\overline{\partial}_{[}a_1\phi_{a_2}\cdots_{a_{2k}]}. \end{aligned}$$

While for the verification of the first $(K-1)$ equations, in (14), it is sufficient to only use (6) (and (12)) the last equality requires knowledge of how $\psi_{a_1}\cdots_{a_{2k}}$ is solved in terms of $\psi_{a_1}\cdots_{a_{2k-2}}$, i.e. $(9)_{k=K}$. While the square integrability of $\Psi$ presumably implies $\|q\Phi\| < \infty$ (via (13)), the discussion of whether $Q^\dagger\Psi = 0$ actually implies $\|\Phi\| < \infty$ (when $\Phi$ is defined via (13)) will be more complicated (but could perhaps roughly go as follows: $Q^\dagger\Psi = 0$, resp. $Q\Psi = 0$, implies that the first derivatives of the $\Psi^{(n)}$, divided by $q$, are square-integrable, which is in conflict with the normalizability of $\Psi$, unless $\Psi(q=0) = 0$ – which probably improves the behavior of $\Phi$ at $q = 0$ such that $\|\Phi\| < \infty$, when $\|q\Phi\| < \infty$).
If $\Psi = Q^\dagger\Phi$, with $\|\Phi\| < \infty$, $\Psi$ can not be annihilated by $Q$, due to the direct sum decomposition of $\mathcal{H}_+$ into $Q\mathcal{H}_-$, $Q^\dagger\mathcal{H}_-$, and states annihilated by both $Q$ and $Q^\dagger$ (I am grateful to J. Fröhlich for pointing out to me this simple but important fact, which suggested to check whether the solutions (7)-(11) arise as images of $Q$, resp. $Q^\dagger$).

Consider now the general case,

$$\begin{aligned} Q_\beta &= D_a^{(\beta)}\partial_{\lambda_a} + M_a^{(\beta)}\lambda_a \\ Q_\beta^\dagger &= D_a^{(\beta)\dagger}\lambda_a + M_a^{(\beta)\dagger}\partial_{\lambda_a} \end{aligned} \tag{15}$$

In $D = 11$ ($\Gamma^j = \Gamma^{j\dagger} = -\Gamma^{jtr},\ j = 1\cdots 7$) or $D = 4(\Gamma^j \to 0,\ x_j \to 0)$ :

$$\begin{aligned} D_{\alpha A}^{(\beta)} &= \delta_{\alpha\beta}\, 2\partial_A - if_{ABC}\, x_{jB}\overline{z}_C\Gamma^j_{\alpha\beta} \qquad (16) \\ M_{\alpha A}^{(\beta)} &= \delta_{\alpha\beta}\, iq_A + i\Gamma^j_{\alpha\beta}\frac{\partial}{\partial x_{jA}} - \frac{1}{2}f_{ABC}x_{jB}x_{kc}\Gamma^{jk}_{\alpha\beta}\ , \end{aligned}$$

where $A = 1\cdots N^2-1, \alpha\beta = 1\cdots 8$ $(D = 11)\Gamma^{jk} = \frac{1}{2}\ [\Gamma_j, \Gamma_k]$. They satisfy [7]

$$\{Q_\beta, Q_{\beta'}\} = 2i\delta_{\beta\beta'}\overline{z}_E J_E \qquad \{Q_\beta, Q^\dagger_{\beta'}\} = \delta_{\beta\beta'}H + 2\Gamma^j_{\beta\beta'}x_{jE}J_E \tag{17}$$

with

$$J_E = -if_{EAA'}(x_{jA}\partial x_{jA'} + z_A\partial_{A'} + \overline{z}_A\overline{\partial}_{A'} + \lambda_{\alpha A}\partial_{\lambda_{\alpha A'}}) = L_E + S_E \tag{18}$$

and

$$H = (-\triangle + V) - 2if_{EAA'}x_{jE}\Gamma^j_{\alpha\alpha'}\lambda_{\alpha A}\partial_{\lambda_{\alpha' A'}} + f_{EAA'}z_E\lambda_{\alpha A}\lambda_{\alpha A'} + f_{EAA'}z_E\partial_{\lambda_{\alpha A'}}\partial_{\lambda_{\alpha A}} \tag{19}$$

where $\triangle = 4\partial_A\overline{\partial}_A + \partial_{jA}\partial_{jA}$ and $V = q^2 + \tilde{V}$ being twice $V(x, \frac{1}{\sqrt{2}}Z, \frac{1}{\sqrt{2}}\overline{Z})$ given in e.g. (4.11) of [7]. The superalgebra (15) alone (!) implies the following (commutation) relations $((\beta\beta')$ denoting symmetrisation $(\frac{1}{2}\beta\beta' + \frac{1}{2}\beta'\beta))$:

$$\begin{aligned} \left[D_a^{(\beta}, D_{a'}^{\beta')}\right] &= 0 \qquad (20) \\ \left[M_a^{(\beta}, M_{a'}^{\beta')}\right] &= 0 \qquad (21) \\ D_a^{(\beta}M_a^{\beta')} &= i\delta_{\beta\beta'}\overline{z}_E L_E \qquad (22) \\ \left[M_{\alpha A}^{(\beta}, D_{\alpha' A'}^{\beta')}\right] &= \delta_{\alpha\alpha'}\delta_{\beta\beta'}\overline{z}_E f_{EAA'} \qquad (23) \end{aligned}$$

and, using also the specific form of $H$, (19),

$$\begin{aligned} \left[M_{[\alpha A}^{\beta}, D_{\alpha' A']}^{\dagger\beta'}\right] &= \delta_{\alpha\alpha'}\delta_{\beta\beta'}z_E f_{EAA'} \qquad (24) \\ D_a^\beta D_a^{\beta'\dagger} + M_a^{\dagger\beta'}M_a^\beta &= \delta_{\beta\beta'}(-\triangle + V) + 2\Gamma^j_{\beta\beta'}x_{jE}L_E \qquad (25) \end{aligned}$$

$$\left[M_{\alpha A}^\beta, M_{\alpha' A'}^{\beta'\dagger}\right] + \left[D_{\alpha A}^{\beta'\dagger}, D_{\alpha' A'}^\beta\right] = -2ix_{jE}f_{EAA'}(\delta_{\beta\beta'}\Gamma^j_{\alpha\alpha'} + \delta_{\alpha\alpha'}\Gamma^j_{\beta\beta'})\ . \tag{26}$$

The equations $Q^{(\beta)}\Psi = 0, Q^{(\beta)\dagger}\Psi = 0$, read:

$$\begin{aligned} (2k-1)M_{[a_1}^{(\beta)}\psi_{a_2}\cdots_{a_{2k-1}]} &= D_{a_{2k}}^{(\beta)}\psi_{a_1}\cdots_{a_{2k}} \qquad (27) \\ k &= 1, \cdots, K \\ (2k-1)D_{[a_1}^{(\beta)\dagger}\psi_{a_2}\cdots_{a_{2k-1}]} &= M_{a_{2k}}^{(\beta)\dagger}\psi_{a_1}\cdots_{a_{2k}}\ . \qquad (28) \end{aligned}$$

Due to the non-commutativity of $M$ with $M^\dagger$ they are slightly more difficult to solve (but also less singular, as $\vec{M}^\dagger\vec{M} > 0$). The solution of the first equations, $M_a\psi = D_b\psi_{ab}, D_a^\dagger\psi = M_b^\dagger\psi_{ab}$, are $\psi = (M^\dagger M)^{-1}M_a^\dagger D_b\psi_{ab}$ and (with $\psi_{abc}$, totally anitsymmetric, arbitrary)

$$\psi_{ab} = 2D^\dagger_{[a}M_{b]}(M^\dagger M)^{-1}\psi + M_c^\dagger\psi_{abc}\ . \tag{29}$$

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